INNOVATIONS IN WOODWORKING INDUSTRY AND ENGINEERING DESIGN

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Article

GRAPHO-ANALYTICAL APPROACH FOR OBTAINING THE UNFOLDINGS OF NON–EXTENDABLE SURFACES OF THE SECOND AND HIGHER DEGREES OF DETAILS IN THE WOODWORKING AND FURNITURE INDUSTRY

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ABSTRACT

This article presents graphical–analytical methods for approximating the developments of non-developable surfaces for components used in the woodworking and furniture industries. Surface development is a key design stage that converts a three-dimensional object into a two-dimensional cutting pattern. Double-curvature surfaces, such as spheres, ellipsoids, and paraboloids, cannot be developed onto a plane without distortion, tearing, or overlap. They must therefore be approximated by segments of developable cylindrical or conical surfaces. The increasing demand for graphical accuracy and technological precision makes this problem particularly relevant. The aim of the study is to obtain approximate developments of toroidal, spherical, and ellipsoidal surfaces by combining graphical construction with analytical calculations. The method replaces a theoretically non-developable surface with a set of small developable segments that can be unfolded with minimal distortion. This decomposition enables complex shapes to be manufactured from sheet material using conventional rolling or bending processes. Analytical calculations confirm that errors are unavoidable in purely graphical developments because of the approximation itself. Computer-aided design (CAD) reduces these errors by allowing individual surface segments to be constructed at a larger scale rather than drawing the entire complex geometry at once. The approach is applicable to the design and manufacture of industrial tanks, silos, sheet-metal dome roofs, spiral staircases, decorative columns, and transport augers.

Keywords: *graphical–analytical methods; error; arc length; non-developable surfaces; manufacturing; precision*

INTRODUCTION

Products with complex shapes and dimensions are created in many fields, including art, crafts, and industrial production. Such products are composed of simple geometric forms or combinations of them and are often manufactured from sheet material [1, 2]. Their surfaces can be classified as developable or non-developable. Production generally includes three stages: construction of the development, cutting along the developed pattern, and assembly of the component using permanent joints (adhesive bonding, welding, or riveting) or detachable joints

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(screws, studs, or bolts) [3, 4]. In the furniture industry, these components form parts of assemblies, machines, joints, and equipment. Their dimensions must therefore match those of the elements with which they are assembled, requiring high development accuracy [5, 6].

Such accuracy cannot always be achieved using purely graphical methods for developing both developable and non-developable surfaces.

To obtain the development of the developable surfaces whose development is a plane figure (polyhedrical, cylindrical, conical surfaces, cylindroids, conoids), three methods are used: the parallel-line method (for edged bodies, cylinders, cones); the normal-section method (for surfaces in general position); and the triangulation method (for ridged bodies and cones) [7, 8]. For cylindrical surfaces with equation $(x-x_0)^2 + (y-y_0)^2 = R^2$ (the base is in the horizontal plane and the axis is parallel to the z-axis) (and in the general case of any straight rotating cylinder with a base in any projection plane and height (axis) perpendicular to the base) the unfold is a rectangle with a base, the circumference length of the base of the cylinder and a height equal to the height of the cylinder. In the graphical construction of the unfold of the cylinder, it is approximated to the n-wall prism inscribed in it [9]. In this case, an error is obtained because the arc and its corresponding chord have different lengths, ld and lx, respectively:

$$ld = \frac{r.\pi.\varphi^0}{180^0} = 0{,}017453.r.\varphi^0 \quad (1)$$

$$lx = 2.rsin.\left(\frac{\varphi}{2}\right)$$

where:

φ is the central angle, which is $\varphi = 360^0/n$.

n is the number of sides of the inscribed prism, ld>lx.

From the analysis made, it follows that the graphical method always leads to a systematic error, which can be reduced by increasing the number of sides of the inscribed prism, but this is associated with an increase in drawing work, and the error cannot be reduced to zero. This conclusion applies to all cases in which surfaces with a directrix circle, ellipse-elliptical cylinders with an equation are made $x^2/a^2 + y^2/b^2 = 1$, parabola-parabolic cylinders with equation $y^2 = 2px$, Hyperbola -hyperbolic cylinders with equation $x^2/a^2 - y^2/b^2 = 1$, or arcs of a circle. In these cases, the unfold is a rectangle with a base the length of the directrix, respectively, ellipse, parabola, hyperbola, etc., and a height equal to the height of the cylinder [10]. To determine the length of the directrix, it is approximated by a polygon entered in it.

The development of a straight circular (rotating) cone with an equation $x^2/a^2 + y^2/b^2 - z^2/c^2 = 0$ (its bases are parallel to the horizontal plane, and the axis parallel to the z-axis) is a (sector) triangle with one curvilinear side - an arc with a radius equal to the generatrix of the cone $l=\sqrt{[h2+(d/2)2]}$, arc length ld = π.d and angle subtended by the arc $\alpha d=180^0.d/l$. where h is the cone height, and d is the diameter of the cone base.

Non-developments are formed by the movement of a generating curve along a directrix – for example, a toroidal surface is formed by the movement of a circle (generatrix), the center of which moves along a circular directrix – a circle. The triaxial ellipsoid (with equation $x^2/a^2 + y^2/b^2 + z^2/c^2=0$) is formed by the movement of a generating curve ellipse (parallel to itself with variable semi-axis lengths), its plane always remaining perpendicular to the planes of the other two ellipses, and the ends of the axes (of the forming ellipse) always lying on the other two ruling ellipses. If the axes of the forming ellipse are equal to each other, i.e. the generating curve is a circle, a rotational ellipsoid with an equation will be obtained $x^2/a^2 + y^2/a^2 + z^2/c^2=0$, and in a special case, if the axes of all three ellipses are equal to each other, a sphere with an equation will be obtained $(x-x_0)^2 + (y-y_0)^2 + (z–z_0)^2=R^2$.

In a similar way, other second- and higher-order surfaces are obtained, for example, one-sheet hyperboloid with an equation $x^2/a^2 + y^2/b^2 - z^2/c^2=1$; two-sheet hyperboloid with equation $x^2/a^2 + y^2/a^2 - z^2/c^2= -1$; elliptical paraboloid with equation $x^2/a^2 + y^2/b^2 = 2pz$; hyperbolic paraboloid with equation $x^2/a^2 - y^2/a^2 = 2pz$ and others. The developments of these surfaces are not plane figures and are obtained by approximating the shape to the development of several developable surfaces inscribed in them.

As can be seen from the equations, these will be curved lines, arcs of circles, etc., i.e., any approximation, and in particular their approximation by straight-line segments, will lead to errors. In the various sectors – industry, energy, hydrotechnical facilities, construction, household products, transport, and others- there are a significant number of products shaped as non-developable surfaces [11,12]. Purely graphical development of non-developable surfaces or their combinations is approximate and may be insufficiently accurate for practical use. It is therefore necessary to apply a graphical–analytical method to obtain developments with high accuracy, combining the merits of graphical development methods and the accuracy of analytical calculations [13].

MATERIALS AND METHODS

To obtain the developments of rotational non-developable surfaces (e.g. sphere, torus), two methods are applied: meridian (wedge, cylindrical) sections and equatorial sections (parallel sections, belts) [9]. Since the methods are known, the issue of applying analytical calculations to obtain an exact, rather than an approximate development of the example of a spherical surface will be considered and analyzed in Fig. 1. On the horizontal projection, the lengths of the arcs ⌒I, ⌒ II and ⌒III and the corresponding chords I, II, III of one sector (wedge) of the sphere lying on three mutually parallel planes parallel to the equatorial horizontal plane, and the frontal projection shows the lengths of the arcs ⌒i, ⌒ii and ⌒iii and the corresponding chords I, II, III lying on the frontal section of the surface of the sphere and whose endpoints lie on the horizontal planes parallel to the equatorial plane. To the right of the two projections of the sphere is built a twelfth part of the development of the sphere, and along the vertical axis are plotted the lengths of the arcs ⌒i, ⌒ii and ⌒iii, and horizontally symmetrically on the axis the lengths of the arcs ⌒I, ⌒ II and ⌒III, respectively.

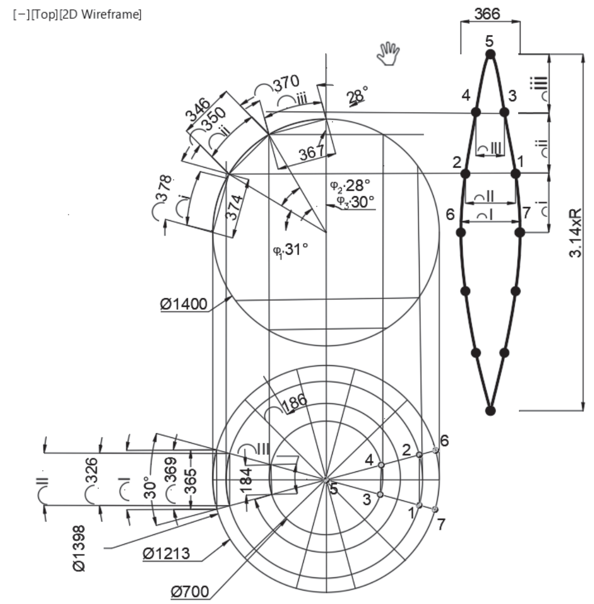

Figure 1. Development of a spherical surface (1/12 of the surrounding surface)

The more accurate value of the arc lengths is obtained using the formulas given above to determine them. The points obtained are joined by the known graphical methods, and the resulting one-twelfth segment of the development will be more accurate and larger in size than the expansion obtained graphically, since the arc lengths are always greater than the corresponding chord lengths with which the circle is approximated: ⌒i >i, ⌒ii>ii, ⌒iii>iii, ⌒I>I, ⌒ II>II and ⌒III>III. The magnitude of the error depends on the number of wedges used to approximate the spherical surface. It decreases with an increase in their number, for example (16), and vice versa, increases with a decrease in their number, for example (8), because an increase in the number decreases the central angle of the sector φ. For 12 sectors φ=30º, for 16 sectors φ=22,5º, for 8 sectors φ=45º. In order to optimize and reduce calculations, it is recommended that the angles in the horizontal and frontal projections be equal φ= φ1 = φ2 = φ3. The error also depends on the radius of the extensible surface, since both radius and angle are involved in the formulas for the length of an arc or chord φ. The full unfold has the form of the full unfold found in a graphic way in Fig. 2.

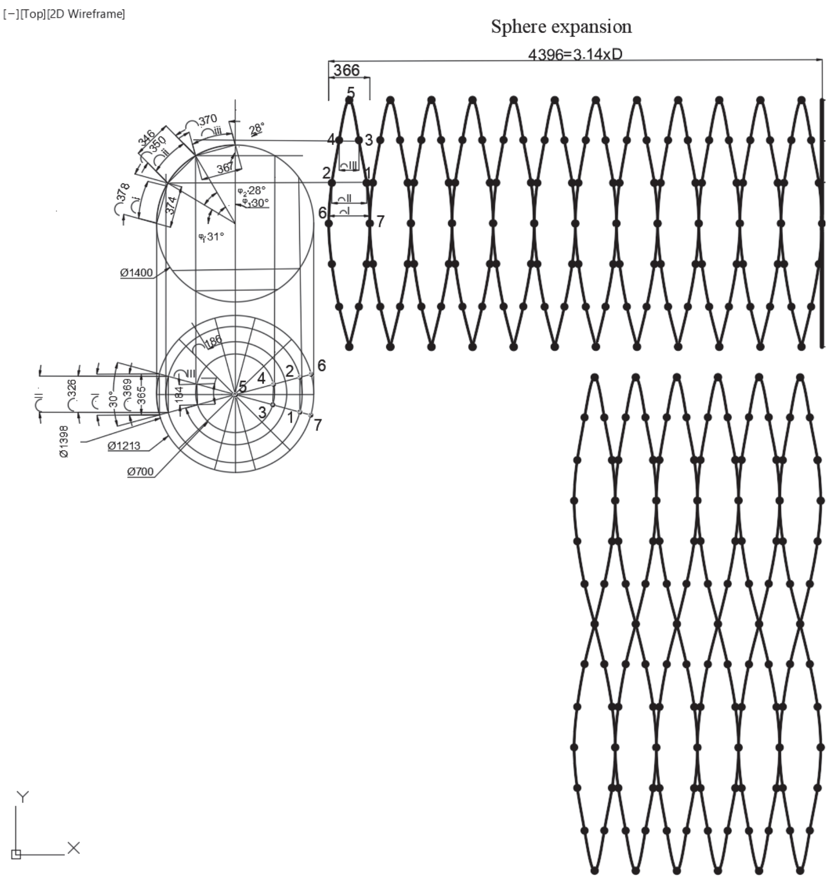

Figure 2. Unfolding a sphere using the method of meridian (wedge, cylindrical) sections

The development of the spherical surface by the graphoanalytic method can also be done by the method of equatorial sections (parallel sections, belts), polyconic method, with the full development having the form of the full development found graphically in Fig. 3. In the grapho-analytic method, the development is obtained by working with the actual dimensions of the arcs of the development surface, in the specific case of the sphere, instead of the approximating chords.

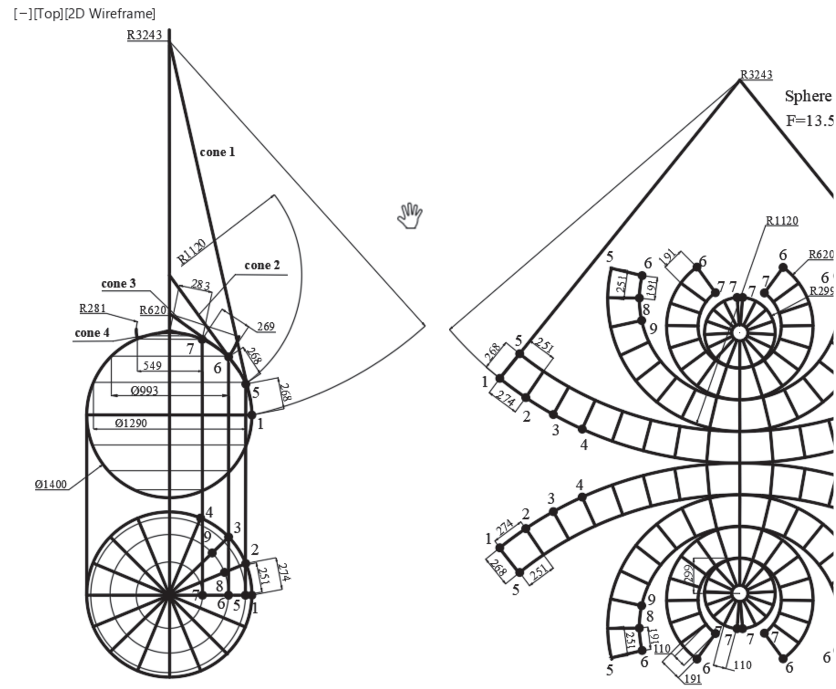

Figure 3. Unfolding a sphere using the method of equatorial sections (parallel sections, belts), the polyconic method

RESULTS AND DISCUSSION

The graphical–analytical method can also be applied to the development of toroidal surfaces, and for simplification, the development of a quarter of the toroidal surface under consideration is found. The rest of the development of the surface is the same as the one found and can be constructed using two Mirror commands in the basic engineering product AutoCAD [14,15]. It is recommended to divide the frontal projection of the torus into equal parts – angular sectors [16]. The developments of all identical angular sectors are obtained in the same way as the development of a sphere is obtained. In this particular case, the method of meridian wedges is applied [17,18]. An element of the unfold is constructed along the horizontal line of the lengths of the arcs, respectively, $\frown$i, $\frown$ii, $\frown$iii, and the intermediate chords between them. To calculate the exact values of the arc lengths, instead of the approximating chords i, ii, iii, and the intermediate ones, the dependence is used:

$$ld = \frac{r.\pi.\gamma}{180^0} \tag{2}$$

where:

R=$r_{\text{throat torus}}$ +$r_{\text{section torus}}$– r_{section} cosγ_i (to R=R_{torus}; recommended γ_i =equal);

after that $R=R_{torus} + r_{section}.\cos\varphi$ (for this specific case $\varphi=30^0$).

The resulting points are joined by the known graphical methods and the resulting one part of the torus unfold (which consists of several sectors in this case five) will be more accurate and larger in size than the unfold obtained graphically, since the arcs are always larger than the chords with which the cross-sectional circle (the forming circle of the torus) is approximated: $\overset{\frown}{I}$ and $\overset{\frown}{i}>i$, $\overset{\frown}{ii}>ii$, $\overset{\frown}{iii}>iii$ including the intermediate ones, which lie on the circles of the meridian sections.

Fig.4 shows the setting of the development of the sectors of a quarter of the torus, and in Fig.5, the development itself.

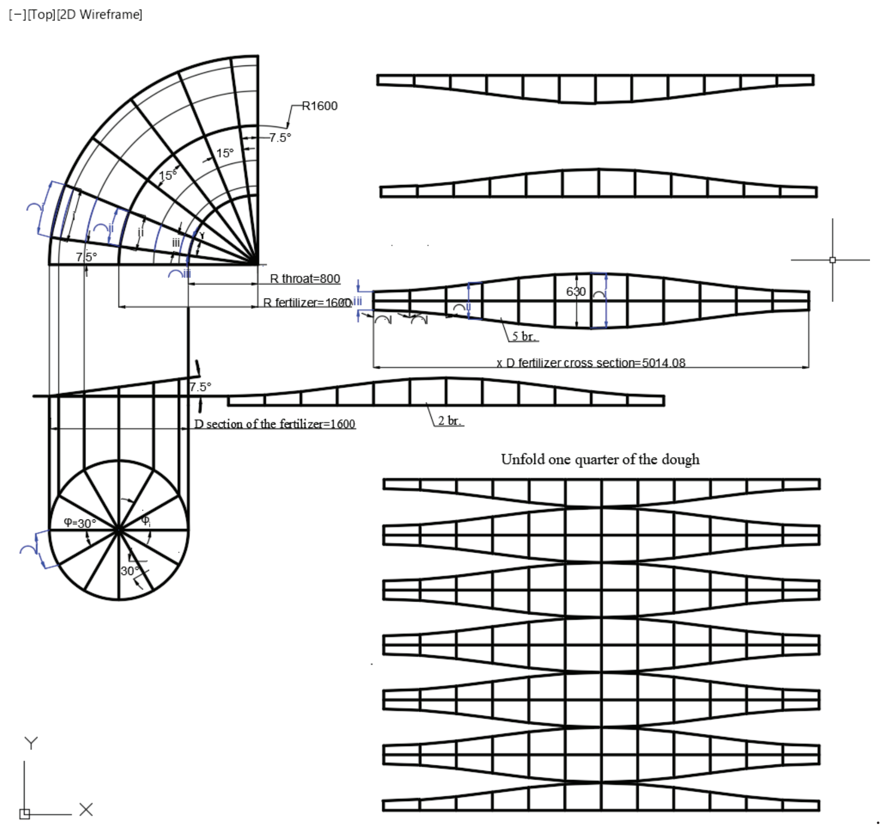

Figure 4. Unfolding the sectors of a quarter of the torus
$\Delta=(n.\pi.r.\ \varphi/180^0 - n.2.r.\ \sin\varphi/2).100/2.\pi.r=n.r(\pi.\varphi/180^0-2.\sin\varphi/2).100/2.\pi.r=n(\pi.\varphi/180^0-2.\sin\varphi/2).50/\pi.$

From the analysis of the dependence, it is found that the error depends on the number of written polygons n in the circle and the central angle of the sector φ, which uniquely depends on n. From the calculations made, it has been found that when inscribed respectively hexagon, octagon, twelfth gon and twenty-four gon, the error is respectively $\Delta=4.67\%$; 2.486%; 1.12%; 0.2139%. Such an error in the diameters leads to the impossibility of joining the dimensions of the elbow (in this case, the torus), for example, with ready-made pipes 48´´ (outer diameter of the pipe 1219 mm) octagon, twelve-rectangon (dodecagon) and twenty-four-quarter (icositragon) the diameters of the knee will be 1162.07mm; 1811.43mm; 1205.34mm; 1216.33mm (smaller by 56.93mm; 30.30mm; 13.65mm; 2.69mm, respectively).

Of particular interest is the construction of the unfold of the helicoids (propeller surfaces). In woodworking and furniture design, the helical surface is one of the most complex geometric shapes to design and implement. The issue of constructing the unfold of a straight helixoid on which all elements moving along the helical line are perpendicular to the axis of the cylindrical surface, cross the axis during the entire movement, and are the same in length. The helical line is defined as the trajectory of a point moving at a uniform rectilinear velocity along the formation of a straight circular cylinder with radius r, R (d, D), which rotates around its axis at a constant angular velocity **ω**. agricultural machinery and equipment, etc., the construction of the development by the graphic method is well known [19]. It is illustrated in Fig.5.

The development is obtained by approximating the circle along the small d and the large diameter D by inscribed hexadecimal angles. In this case, the lengths of the circles will be shorter due to the smaller length of the substitute chords than the corresponding arcs of the angular sectors, and they, the lengths of the circles, participate in determining the lengths of the unfolded h, H, which are calculated by the dependencies:

$$h=\sqrt{(\pi.d)^2+L^2} \text{ for the small diameter}$$
$$H=\sqrt{(\pi.D)^2+L^2} \text{ for the large diameter} \quad (3)$$

where:

L is the pitch of the helicoid. The lengths of the helicoid unfold are involved in calculating the development angle θ.

Therefore, it is recommended to work using the graphical–analytical method. Otherwise, the angle θ will be obtained with a smaller value, and this will reduce the pitch of the helical line L in real execution. The angle θ is determined by the dependence: $\theta=H.360^{o}/\pi.D$ for the helical line along the large diameter; $\theta=h.360^{o}/\pi.d$ for the helical line on the small diameter.

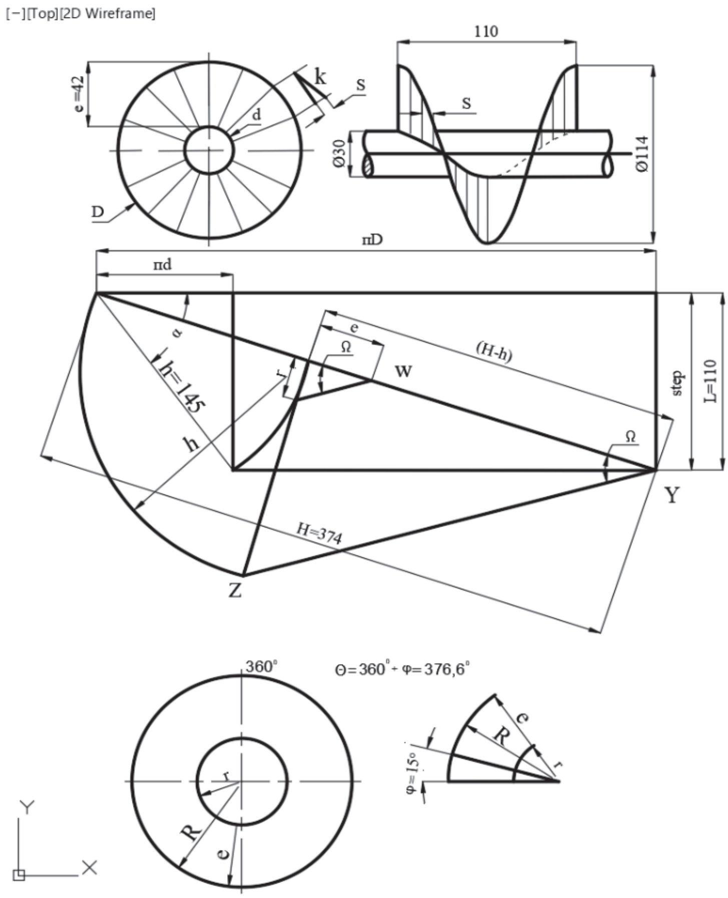

Figure 5. Unfolding a straight helicoid

Helicoids have a specific feature. Since the angle θ is always greater than 360º (H > D and h > d), the unfold must be cut and extended with the complementary part above 360º.

When constructing the extensions of ellipsoids (triaxial, rotational), in order to simplify the work, it is recommended that they be positioned so that the major axis is horizontal [20]. It is recommended that, in order to facilitate work, the rotational axis should be vertical, which allows the polyconic method (of belts, equatorial sections) to be applied. The accuracy of the developments is improved, as there is no need to calculate arcs and approximations [21]. In this way, it is easier to work only with circles or arcs of circles, the number of belts is reduced, and therefore the calculation procedures, which are reduced to determining the development angles, which significantly reduce the number of calculations and improve construction efficiency and accuracy.

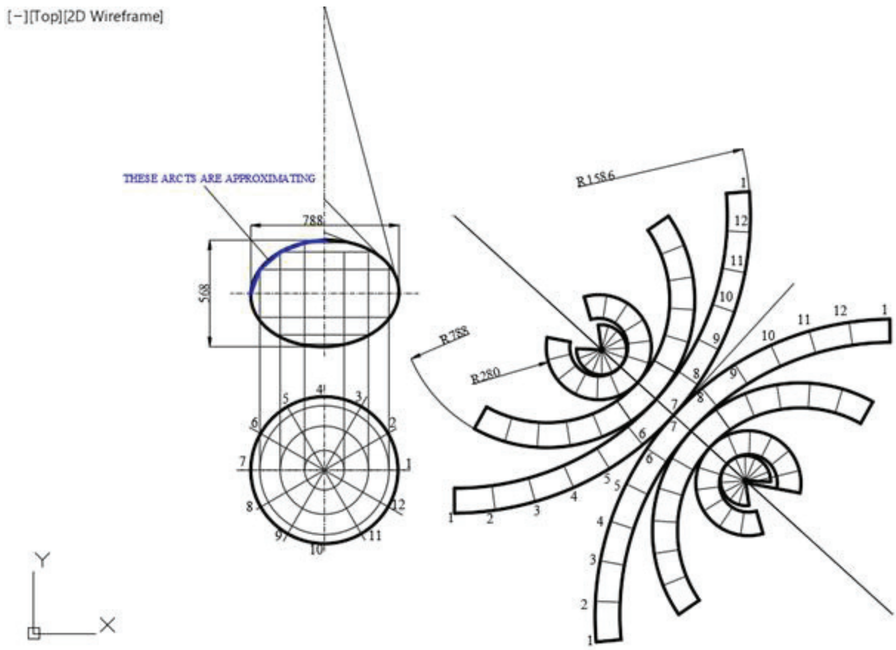

Figure 6. Development of a rotational ellipsoid

CONCLUSIONS

The proposed graphical–analytical approach for generating developments of second- and higher-order surfaces has broad applicability to a wide range of geometric objects. The results provide a basis for a systematic methodology covering specific structural elements, such as complex transition pieces, while ensuring geometric and technological accuracy between different cross-sections. A library of graphical primitives could support intelligent automation in a CAD environment and help translate theoretical models into manufacturing procedures for the modern furniture industry.

The analysis supports the following main conclusions:

- It has been found that when obtaining planar developments of surfaces with double curvature (spherical, elliptical, hyperboloid), approximation by graphical–analytical methods is the necessary approach to compensate for the inevitable deformations (stretching and contraction) caused by the impossibility of isometric representation of these geometries in a plane.
- It has been proven that the proposed method is a highly effective method for approximating theoretically non-developable second- and higher-order surfaces, providing an optimal transition between the complex 3D shape and its 2D projection.

- Compared to classical graphical constructions, the graphical–analytical method significantly reduces the cumulative error. By scaling the individual segments (belts), high accuracy is achieved in determining the geometric parameters (arc length and unfolding angle), which is essential for cutting of sheet material in the furniture industry.
- The results of the study confirm that although the errors in the approximation of objects such as spheres, tori, and paraboloids are mathematically inherent, the graphical–analytical method provides accuracy exceeding the production tolerances in the processing of wood and its derivatives.
- The developed method demonstrates invariance with respect to the complexity of the original surface. This feature allows its smooth implementation in modern CAD/CAM systems for automated design of structural elements with complex geometry.
- Structuring the method creates a theoretical basis for building a specialized library of graphic primitives. It serves as the foundation for the automated generation of standardized developments, shortening the time required for production preparation.
- The possibilities for industrial realization of non-developable shapes are directly determined by the balance between computational complexity and geometric fidelity. The digitized graphical–analytical approach (polyconic and meridian) minimizes technological deviations, ensuring high quality of the finished products with minimal consumption of raw materials.

AUTHOR CONTRIBUTIONS

Conceptualization, A.B.; methodology, A.B.; software, A.B.; validation, A.B.; formal analysis, A.B.; writing—original draft preparation, A.B.; writing—review and editing, A.B.; visualization, A.B. The author has read and agreed to the published version of the manuscript.

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CONFLICTS OF INTEREST

The author declaress no conflicts of interest.

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