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A MODEL OF THE STATICALLY INDETERMINATE TENSION IN THE BLADE OF A HORIZONTAL BANDSAW

Stefan Stefanov[1], Valentin Atanasov[2]
[1] University of Forestry – Sofia, Department of Mechanical Engineering and Automation of Production, stefanov_sh@abv.bg
[2] University of Forestry – Sofia, Department of Woodworking Machines, vatanasov_2000@yahoo.com

ABSTRACT

An original model is proposed for determination of the statically indeterminate tension (internal tensile force) in the blade (saw band) of a horizontal bandsaw with a tensioning spring mechanism. The statically indeterminate problem is solved in a way missing in the bandsaw references but required by the Strength of Materials science: based on a deformation compatibility equation. In the case considered, in that equation the spring parameters are also involved. The model proposed is a generalization of a published previous model (Stefanov 2013) regarding a vertical bandsaw blade tension.

Key words: bandsaw blade, horizontal bandsaw, spring mechanism, statically indeterminate tension

INTRODUCTION

Based on a previous paper (Stefanov 2013), this paper represents a study aimed at establishing determination of the internal tensile force in a bandsaw blade in a way the Strength of Materials science requires. It is meant that the saw band, the two wheels and the spring mechanism for tensioning the saw band build *a statically indeterminate system*. The half of the contents of every Strength of Materials course, including (Stefanov 2007), is devoted to statically indeterminate systems while this issue is not brought to light in the bandsaw references. Separately, the *closed contour of the band is a bright engineering example of internal statical indeterminacy* (Stefanov 2007) as discussed below.

In such a case, *a deformation compatibility condition should be formulated for solving the statically indeterminate problem*. In (Stefanov 2013), such a compatibility equation has been set which is valid for a vertical bandsaw with spring tensioning the blade.

1. CALCULATION SCHEME, COMPATIBILITY CONDITION AND *X*

An original scheme is shown in Fig. 1. An instantaneous kinetostatic position of the saw band is illustrated. The blade moves counterclockwise with a constant velocity $v = \omega R = constant$. On the center B of the left, driven wheel, a force N_B of the spring is exerted. It is transmitted to the band as a transversely distributed load of intensity q_{N_B} (force per unit length). In the model considered, q_{N_B} appears intermittently at H, remains constant and again intermittently disappears at G. This is an idealization entailed by the assumption that the spans (branches) GD and HC are ideally straight and transfer into exact left and right half-circumferences. In fact, the Euler postulation of an ideally flexible and massless cord girdled a wheel is accepted as valid for a saw band (and for belt drives, as well).

In reality, the blade is not ideally flexible but has certain bending stiffness. To find out the actual smooth variation of the saw-band's curvature and of the force distribution intensities is not an easy problem at all. It will concern additional deformation, strength, dynamic and vibration effects (Csanady & Magoss 2011, Wagenführ und Scholz 2007). They could be involved in further expanding the model. As for now, the first, basis-laying thing to do is deduction of the equations for the basic, nominal internal loads along the saw band based on solving the statically indeterminate problem.

From the q_{N_B} intensity, the imaginarily concentrated force $N_B = q_{N_B}.2R$ results. The right, driving wheel presses the band with a distributed transverse force with intensity $q_{N_A}(\varphi)$ concentrated imaginarily into a force N_A. The band is moved by a tangentially distributed friction force with an intensity $\tau(\varphi)$. By concentrating $\tau(\varphi)$, an imaginary single force T (of traction) appears as a resultant at some distance f from the center A. The equation $T = \mu N_A$ holds where μ is a working coefficient of friction (coherence). The ratio $T/N_A = \mu$ is also tgα (Fig. 1). Along the left and the right half-circumference, an equal intensity $q_\Phi = constant$ of two transversely distributed centrifugal inertial forces $\Phi = q_\Phi.2R$ is introduced. Besides, a longitudinally distributed cutting force of intensity p acts along a segment EF of the cutting (the lower) branch of the band. From p, an imaginary concentrated force P results. For simplification, $p = constant$ can be introduced, i.e. $p = P/h_p$ where h_p is the length of the EF segment (the height of the cut).

The cutting span of the blade is guided by rollers of guides F_1 and E_2. The right guide is immovable: $l_2 = constant$. The left one is movable: the length l_1 is set according to h_p. Forces P_1 and P_2 from the blade guides displace the cutting spam EF by w. This displacement, and correspondingly the angles α_1 and α_2, should be small enough as to cause slight blade bends. Otherwise, great w, α_1 and α_2 would make the blade bend at F_1 and E_2 with the roller's radius which is much shorter than R of the two wheels. This would cause bending stresses much higher than those of the blade bending round the wheels. Eventually, the angles α_1 and α_2 are usually much less than 2°. They are shown exaggerated (about 6°) in Fig. 1 for better illustration and easier distinguishing.

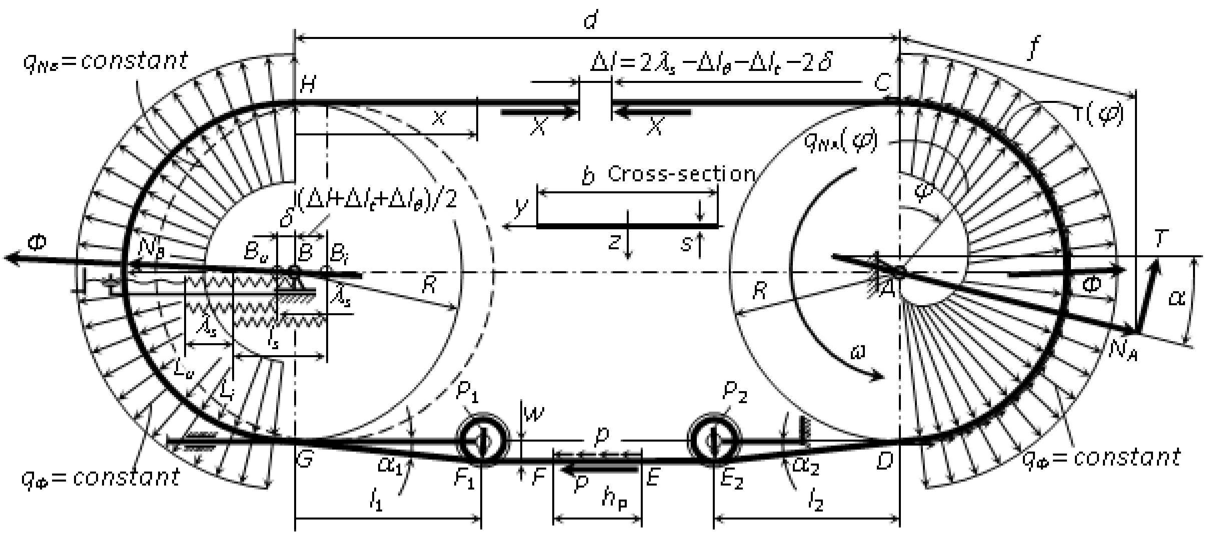

Figure 1: Calculation scheme

Thus, the forces N_B and Φ, exaggeratedly slanted in Fig. 1, are practically horizontal. The forces P_1 and P_2 are practically vertical and very little comparatively to X. Indeed, let the equilibrium of a blade part at F_1

be considered (Fig. 2) where the two forces X and the force P_1 act. The force equilibrium condition in the P_1 direction is $P_1 = 2X\sin(\alpha_1/2)$. For an actual angle $\alpha_1 < 2^o$ it turns out that $P_1 < 0{,}035X$, i.e. P_1 is smaller than 3,5% of X. Hence, P_1 and P_2 can be neglected in the equations, which follow below, of the kinetostatic equilibrium of external forces on the blade. Only the rest of the forces in Fig. 1 will be involved, and among them N_B and the two Φ forces will be considered as horizontal. In fact, X will be determined below for $w = 0$, and the result will be valid also for $w > 0$ at a relative error less than 5 %.

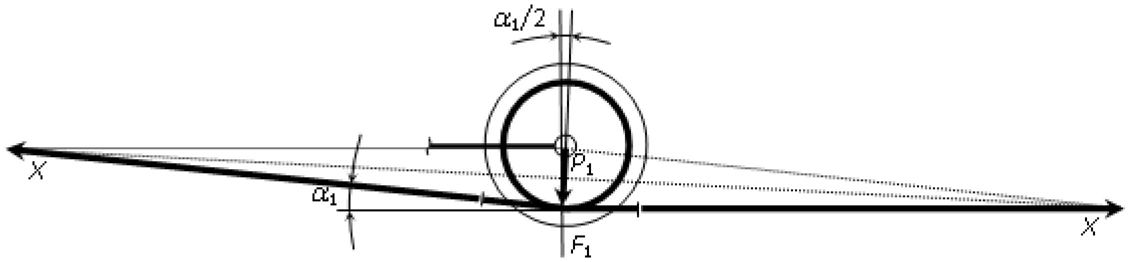

Figure 2: The forces X and P1

In Fig. 1, the moment equilibrium condition $\Sigma M_{i,A} = 0$ leads to $PR = Tf$. From the condition $\Sigma H_i = 0$ for the horizontal, the equation $N_A\cos\alpha + T\sin\alpha = P + N_B$ results. Fulfilling the condition $\Sigma V_i = 0$ for the vertical forces forces means $T/N_A = \mathrm{tg}\alpha$. Provided that P is a given force, the equilibrium equations have involved the following five loading parameters: T, f, N_A, α and N_B. Further below, another equation will yield $\mu = \mathrm{tg}\alpha$. Thus, four parameters remain to be determined under three equilibrium equations: the problem is one time statically indeterminate.

Consideration of deformation components from the different influences follows, as well as their combination in a compatibility condition for solving the statically indeterminate problem. The order of superposing the influences does not affect the result and therefore a sequence can be chosen which makes the analysis easier. First, let no influences act on the saw band (no forces, nor any tilt of the driven wheel, nor heating). The driven (left) wheel's center is in an initial rightmost position B_i such that the saw band has no lengthening yet.

The blade-length is formed by the two straight-span lengths d and the two half-circumference-lengths πR: $l = 2d + 2\pi R$. Of course, the bottom "straight" span is not exactly straight due to the blade guides. However, its length is practically d again. Indeed, if taking into account the angles α_1 and α_2, the bottom-span length accepts $l_1/\cos\alpha_1$ and $l_2/\cos\alpha_2$ instead of l_1 and l_2. If a maximum of 2^o is taken for α_1 and α_2, then $l_1/\cos\alpha_1 = 1{,}006l_1$ and $l_2/\cos\alpha_2 = 1{,}006l_2$. Thus, insignificant deviations from l_1 and l_2 occur. Respectively, the bottom-span length deviation from d will also be insignificant (at the fourth significant figure).

First, let the influence of a temperature change Δt is considered: $\Delta t > 0$ is heating, and $\Delta t < 0$ is cooling. A free change in the band's length occurs: $\Delta l_t = \alpha_t l \Delta t$ (Stefanov 2007) where: α_t is the coefficient of thermal expansion (or compression if $\Delta t < 0$) per 1°C. In the case of heating, the band lengthens by Δl_t and the left semi-circumference is moved by the spring to the left. Correspondingly, the left wheel centre B_i undergoes a free thermal displacement to the left. What is this displacement in comparison with Δl_t?

The answer is simple: *each displacement of the left-wheel center to the left means lengthening* Δl *of the blade that is twice as long*. Indeed, this concerns the following easy geometric problem: since $2\pi R$ is kept, then $\Delta(2d + 2\pi R) = \Delta(2d) + 0 = 2\Delta d$ where Δd is namely the mentioned displacement of the left-wheel center to the left. Thus, the free thermal move of B_i to the left is $\Delta l_t/2$.

Let the next influence be the appearance of all the forces counted above. They all cause an elastic lengthening Δl_{el} which can be shortly labeled as Δl only. Following this Δl, the left-wheel center moves additionally to the left by $\Delta l/2$ (all the displacements in Fig. 1 have been drawn exaggeratedly large to be seen easier). The total displacement of the heel's center becomes $(\Delta l + \Delta l_t)/2$.

In the model in (Stefanov 2013), no additional blade lengthening Δl_θ has been introduced as owing to an angle θ of driven-wheel tilt. The abstention from Δl_θ has been motivated for it less influences X and for it is somehow treated in different and not comparable ways in the references. And yet, in the proposed generalization of the model, Δl_θ is introduced: see Fig. 3.

A top view of the upper half of the saw band from Fig. 1 is shown in Fig. 3. If the driven (left) wheel is not tilted and the blade trends to slip out laterally in the direction from L_2' to L_1', then the friction along L_1'L_2' could only counteract. Otherwise, thanks to a tilt of the wheel, the force N_B from Fig. 1 yields in Fig. 3 a lateral component in the direction from L_1' to L_2'. That force component additionally stabilizes the blade against a lateral slip-out.

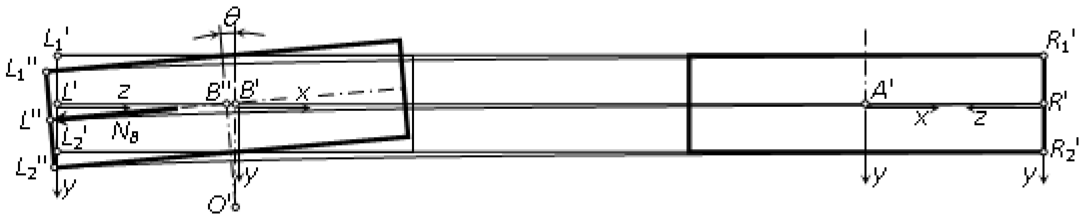

Figure 3: Tilt of the driven wheel

The angle θ should be as small as not to case an opposite effect: slipping the blade out in the direction of the formed lateral N_B component. Besides, θ causes bending and torsion of the blade, as discussed below, and the corresponding stresses should not be excessive. Thus, θ is restricted under 1°. It is shown in Fig. 3 as exaggerated as to see better the effect caused. Although θ is small, it entails some displacement $\lambda_\theta = \overline{B'B''}$ of the wheel's center. With the small θ, the equation $\lambda_\theta = \theta.\overline{B'O'}$ is valid where O' is the intersection point of the axes of the tilted and not tilted wheel's positions. Analogous equation for λ_θ is also set in (Eschler 1982).

According to the above statement that each displacement of the left-wheel center to the left means a lengthening twice as much, $\Delta l_\theta = 2\lambda_\theta$ applies. If the point O' in Fig. 3 (i.e. the bearing) is on the other side of the wheel, then, under the same tilt directional sense, Δl_θ will be negative, i.e. shortening. No description of this fact has been found in the references. For the purposes of this paper it is more suitable to have Δl_θ left instead of $2\lambda_\theta$ since in the other references different interpretations are involved. Anyway, Δl_θ can be always worked out. Thus, $\Delta l_\theta/2$ is added to the above expression $(\Delta l + \Delta l_t)/2$ and $(\Delta l + \Delta l_t + \Delta l_\theta)/2$ is obtained.

The tilt of the driven wheel causes bending of (the projection) of the saw band (Fig. 3). This bending is in the *x*-*y* plane and the bending moment is M_z (Stefanov 2007). The great cross-sectional moment of resistance $W_z = b^2s/6$ holds. The central fibre presented by *L*'*R*' in Fig. 3, curving to the position *L*''*R*', is the neutral line. It does not lengthen or shorten with this bending. Whereas, the fibre presented by $L_1'R_1'$ and curving to the position $L_1''R_1'$ lengthens; the fibre presented by $L_2'R_2'$ and curving to the position $L_2''R_2'$ shortens. Thus, under the considered bending and tension caused by θ, the different fibres have resultant lengthenings which differ from Δl_θ, and only the central fibre has Δl_θ.

Up to this point, the displacement $(\Delta l + \Delta l_t + \Delta l_\theta)/2$ of the driven-wheel center has been formed. Further on: the process of coming of the right spring's end from the B_i position to the working *B* position, and the appearance of the spring force equal to N_B, can be presented in the following way. Let first the spring be free and not deformed, and have length of spring l_s. Let the left spring's end be still not caught and pulled to the left, and be in the initial position L_i (while the right spring's end is in the position B_i). Then, let the left spring's end be caught by the screw shown in Fig. 1. It turns and replaces (pulls) the same left end to the left at a distance λ_s (λ of spring) to the utter position L_u. Thus, the right spring's end comes from the position B_i to the working position *B*.

Concomitantly, the working deformation δ of the spring, i.e. the change in its length l_s, should be shown up in Fig. 1. With that, the spring force will be $N_B = c\delta$ where *c* is the spring constant (stiffness). For showing δ up, the right spring's end can be imaginarily unhooked from *B*. It will find itself in the position B_u. Then, δ is the B_uB length, and the B_uB_i length is again λ_s. Hence, Fig. 1 gives the following compatibility equation: $(\Delta l + \Delta l_t + \Delta l_\theta)/2 + \delta = \lambda_s$. The same equation (without involving Δl_θ) has also been worked out in (Stefanov 2013) for a vertical bandsaw.

As already said, the closed band's contour, separately considered, is internally statically indeterminate: a single cross-section cannot divide it into two parts for finding the internal forces in that cross-section from the equilibrium conditions of any of the two free-body diagrams (Stefanov 2007). Thus, the internal forces in question remain statically indeterminate and, therefore, the statical indeterminacy is called internal. For solving the problem and finding the internal forces in a closed contour, it "is opened" first. In the case considered (Fig. 1), an „opening" is already shown in the non-cutting (the upper) span of the band: two infinitely close cross-sections are imaginarily parted from each other and "are allowed" to draw wide apart. However, the two equal and opposite internal tensile forces N_x of the common magnitude *X* are set. This statically indeterminate *X* should be determined in a way as to have the two cross-section connected back.

The above compatibility equation $(\Delta l + \Delta l_t + \Delta l_\theta)/2 + \delta = \lambda_s$ can be multiplied by two and written with the elastic lengthening Δl on the left side. Thus, the condition searched for solving the statically indeterminate problem of the closed contour appears in the following form (more typical for Strength of Materials):

$$\Delta l = 2\lambda_s - \Delta l_\theta - \Delta l_t - 2\delta. \qquad (0)$$

Respectively, Eq. (0) can be interpreted in the following way. First, the two cross-sections of the "opening" are free to draw wide apart. The spring is imagined to be ideally rigid and thus pulling out its left end by $2\lambda_s$ causes the distance $2\lambda_s$ in Eq. (0). Next, the lengthenings Δl_θ and Δl_t occur and thus the two cross-sections draw closer: Δl_θ and Δl_t

are subtracted from $2\lambda_s$. Next, the spring becomes soft, lengthens by δ, and the two cross-sections draw additionally closer by 2δ. To have the "gap" between the two cross-sections completely closed, the two X forces, together with the tensile forces N_x in all the other cross-sections of the band, cause the elastic lengthening Δl. After all, $2\lambda_s - \Delta l_\theta - \Delta l_t - 2\delta - \Delta l = 0$ is obtained, i.e. Eq. (0) with Δl written on the left.

2. DETERMINATION OF *X* AND OF THE INTERNAL TENSILE FORCE $N_X(X)$ ALONG THE WHOLE BLADE'S LENGTH

In Fig. 4, from the circumferential segment *CD* of the band, an infinitesimal portion is detached by two cuts at abscissae x and $x + \mathrm{d}x$. The portion has a length $\mathrm{d}x$ and angular scope $\mathrm{d}\varphi$. Its free-body diagram includes the two internal tensile forces N_x and $N_x + \mathrm{d}N_x$, and the two equal internal bending moments M_y. Along the length of the portion $\mathrm{d}x$, the following forces act: a normal compression force $\mathrm{d}N \equiv \mathrm{d}N_A$, a friction force $\mathrm{d}T$ and a centrifugal inertial force $\mathrm{d}\Phi$. Since $\mathrm{d}x$ is an infinitesimal length, the forces $\mathrm{d}N$, $\mathrm{d}T$ и $\mathrm{d}\Phi$ can be illustrated anywhere along $\mathrm{d}x$, e.g. at the upper end of the portion.

The force equilibrium equation in the radial direction at the angle φ is: $\mathrm{d}N = (N_x + \mathrm{d}N_x)\sin\mathrm{d}\varphi - \mathrm{d}\Phi = (N_x + \mathrm{d}N_x)\mathrm{d}\varphi - \mathrm{d}\Phi = N_x\mathrm{d}\varphi - \mathrm{d}\Phi$ (where $\sin\mathrm{d}\varphi \to \mathrm{d}\varphi$, and $\mathrm{d}N_x\mathrm{d}\varphi$ is a negligible infinitesimal of second order). The force equilibrium equation in the tangential direction is: $N_x = -\mathrm{d}T + (N_x + \mathrm{d}N_x)\cos\mathrm{d}\varphi$. Since $\cos\mathrm{d}\varphi \to 1$, then $\cancel{N_x} = -\mathrm{d}T + \cancel{N_x} + \mathrm{d}N_x$, and the equation $\mathrm{d}N_x = \mathrm{d}T$ results. In it, $\mathrm{d}T = \mu\mathrm{d}N$ can be substituted: the ratio $T/N_A = \mu = \mathrm{tg}\alpha$ is kept as a ratio $\mathrm{d}T/\mathrm{d}N$, as well. Thus, $\mathrm{d}N_x = \mu\mathrm{d}N$ results. After substituting $\mathrm{d}N$ from above, $\mathrm{d}N_x = \mu(N_x\mathrm{d}\varphi - \mathrm{d}\Phi)$ is obtained.

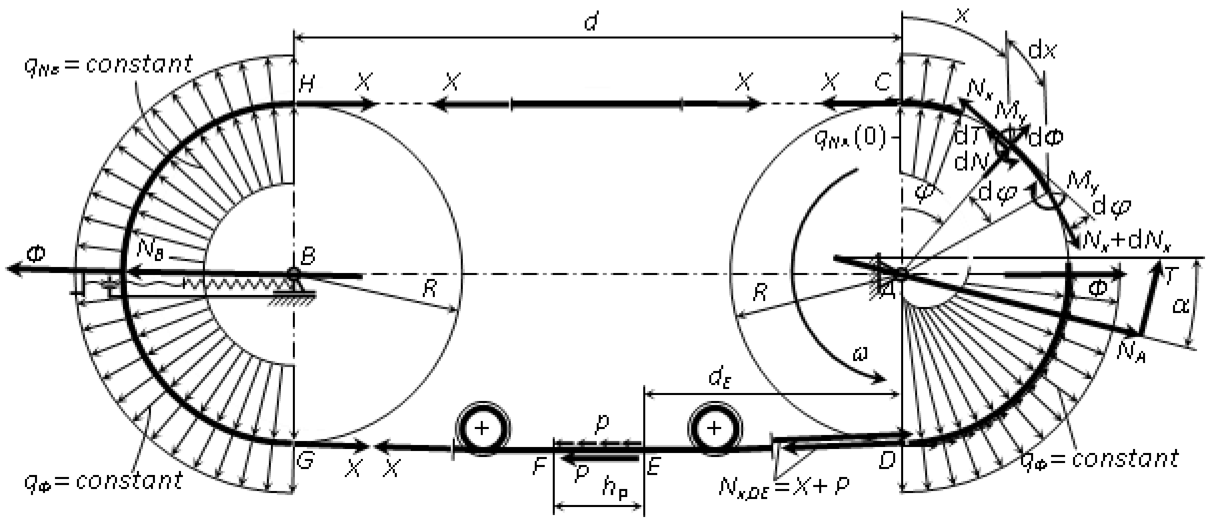

Figure 4: Illustration to the differential analysis

As for $\mathrm{d}\Phi$, the portion's mass is $\rho bs.\mathrm{d}x$ where ρ is the mass density of the material, and $bs.\mathrm{d}x$ is the portion's volume. Then, $\mathrm{d}\Phi = \rho bs.\mathrm{d}x.v^2/R = \rho bs\cancel{R}\mathrm{d}\varphi.v^2/\cancel{R} = \rho bsv^2\mathrm{d}\varphi$. By the way, the q_Φ intensity is $q_\Phi = \mathrm{d}\Phi/\mathrm{d}x = \rho bsv^2/R$, and hence $\Phi = q_\Phi 2R = 2\rho bsv^2$. The above equation $\mathrm{d}N_x = \mu(N_x\mathrm{d}\varphi - \mathrm{d}\Phi)$ takes the form $\mathrm{d}N_x = \mu\mathrm{d}\varphi(N_x - \rho bsv^2)$. It is equivalent to write $\mathrm{d}(N_x - \Phi/2) = \mu\mathrm{d}\varphi(N_x - \Phi/2)$, and thus $\mathrm{d}(N_x - \Phi/2)/(N_x - \Phi/2) = \mu\mathrm{d}\varphi$ applies.

The definite integral of the right-hand side of the last differential equation, from the lower limit $\varphi = 0$ to the upper limit $\varphi = x/R$, yields $\mu x/R$. The indefinite integral of the left-hand side is $\ln(N_x - \Phi/2)$. In agreement with φ from 0 to $\varphi = x/R$, the definite integral should be taken from $X - \Phi/2$ to $N_x - \Phi/2$: $\ln(N_x - \Phi/2) - \ln(X - \Phi/2) = \mu x/R$. Hence, along the segment *CD*, the equation $N_x(x) = e^{\mu x/R}(X - \Phi/2) + \Phi/2$ applies. As for the other band's segments, Fig. 4 shows that $N_x = X$ in the non-cutting span remains to act the same along the whole length *FGHC*. And, for the segment *EF*, the equation $N_x(x) = X + p(h_p - x) = X + (P/h_p)(h_p - x)$ is valid. With that, $N_x(0) = X + P = N_{x,DE}$ where $N_{x,DE}$ comes from the segment *CD* as $N_{x,DE} = e^{\mu\pi}(X - \Phi/2) + \Phi/2$. Hence,

$$\mu = \mathrm{tg}\alpha = \frac{1}{\pi}\ln\left(1 + \frac{P}{X - \Phi/2}\right) \tag{1}$$

where $\Phi/2 = \rho bsv^2$.

A systematic arrangement of the $N_x(x)$ equations along the segments follows.

$$CD{:}\ N_x(x) = e^{\mu x/R}(X - \Phi/2) + \Phi/2 \tag{2}$$

$$DE{:}\ N_x = const = X + P \tag{3}$$

$$EF{:}\ N_x(x) = X + (P/h_p)(h_p - x), \tag{4}$$

$$FGHC{:}\ N_x = const = X \tag{5}$$

Now, Δl can be formed as a sum of components from the segments *CD*, *DE*, *EF* and *FGHC*. The sum will be substituted in Eq. (0). The Δl components from the segments *DE* and *FGHC* where $N_x = constant = X$ are directly substituted with expressions in the form "N_x.(length of segment)/(Ebs)". The Δl components from the segments *CD* and *EF* where $N_x \neq constant$ take the form "[integral of $N_x(x)\mathrm{d}x$]/(Ebs)" (Stefanov 2007). To solve the corresponding two integrals is easy: exponential and linear functions are involved. Besides, $\delta = N_B/c$ should also be substituted in Eq. (0) to involve the spring constant c instead of δ. In addition, N_B is to be substituted from the equilibrium equation of the forces on the left wheel (Fig. 4): $N_B = 2X - \Phi$.

After all the substitutions and mathematical deductions, the following equation results:

$$X = \frac{1}{2d + \pi R + \dfrac{4Ebs}{c}}\left[-P\left(d_E + \frac{h_p}{2}\right) - \frac{PR}{\mu} - \frac{\pi R\Phi}{2} + 2Ebs\lambda_s - Ebs\Delta l_\theta - Ebsl\alpha_t\Delta t + \frac{2Ebs\Phi}{c}\right] \tag{6}$$

where $\Phi = 2\rho bsv^2$.

Having Eqs. (6) and (1), the internal tensile forces N_x along all the band's segments are already determinable by their Eqs. (2) – (5). As well, $N_B = 2X - \Phi$ can be computed. In addition, according to the equilibrium equations (in chapter 1 above), the following equations for N_A, T, and the moment arm f of the T force (Fig. 1) are obtained: $N_A = (P + N_B)\cos\alpha$ where $\alpha = \mathrm{arctg}\mu$, $T = (P + N_B)\sin\alpha$, and $f = PR/T$.

CONCLUSION

The deformation compatibility Eq. (0) has been formulated for a horizontal bandsaw with a spring mechanism for tensioning the blade, and the statically indeterminate problem has been solved as the Strength of Materials science requires. This should be presented in the foreground in the books treating the internal forces and stresses in a saw band. It is seen that a non-simple mathematical expression, Eq. (6), is obtained for X and is carried into Eqs. (2) – (5) for the internal tensile forces along the band's segments. Any different way for determination of the tensile forces in question, based on any combination of components that are not associated with the compatibility Eq. (0), would be groundless.

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